

ARBITRAGE-FREE PRICING DYNAMICS OF INTEREST-RATE GUARANTEES BASED ON THE UTILITY INDIFFERENCE METHOD

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ABSTRACT. We consider the problem of utility indifference pricing of a put option written on a non-tradeable asset, where we can hedge in a correlated asset. The dynamics are assumed to be a two-dimensional geometric Brownian motion, and we suppose that the issuer of the option have exponential risk preferences. We prove that the indifference price dynamics is a martingale with respect to an equivalent martingale measure (EMM) Q after discounting, implying that it is arbitrage-free. Moreover, we provide a representation of the residual risk remaining after using the optimal utility-based trading strategy as the hedge.

Our motivation for this study comes from pricing interest-rate guarantees, which are products usually offered by companies managing pension funds. In certain market situations the life company cannot hedge perfectly the guarantee, and needs to resort to sub-optimal replication strategies. We argue that utility indifference pricing is a suitable method for analysing such cases.

We provide some numerical examples giving insight into how the prices depend on the correlation between the tradeable and non-tradeable asset, and we demonstrate that negative correlation is advantageous, in the sense that the hedging costs become less than with positive correlation, and that the residual risk has lower volatility. Thus, if the insurance company can hedge in assets negatively correlated with the pension fund, they may offer cheaper prices with lower Value-at-Risk measures on the residual risk.

1. INTRODUCTION

Life companies managing pension funds typically guarantee a minimum rate of return on the fund, which is equivalent to issuing a put option. The life company faces a risk of having to raise additional money to cover the deficit if the pension fund fails to achieve the guaranteed rate of return, and will therefore charge a fee to hedge this risk. In a simple situation, the company may buy a put option equivalently structured as the one they have issued, and charge the fee covering the cost of buying this option. The company has then re-insured their risk, and is immune against any losses due to bad performance of

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the pension fund. Many authors have considered the pricing and hedging of interest-rate guarantees using the classical Black & Scholes' option pricing theory, see e.g. Miltersen and Persson [10], and the references therein. Analysis of pricing and hedging of interest-rate guarantees are not only relevant for life companies, but also for other financial institutions offering investment products where there is a guaranteed minimum return.

In this paper we want to analyse the pricing and hedging of interest-rate guarantees in the case when the life company cannot construct perfectly replicating strategies, and therefore is prevented from using the standard Black & Scholes framework. For many life companies it is not possible to cover their risk by simple hedging using similar put options. The pension fund may consist of assets where there exists no options in a secondary market, like for instance mixes of domestic and foreign investments that are hard to hedge, or more illiquid assets like e.g. real estate. Illiquidity may also be an issue for bigger funds with a large market impact due to their size, since they may move the market by their hedging operations. Also, the life span of the pension contract may be so long that there are no possible option contracts available to use for re-insurance. The alternative strategy for the company is then to try to hedge the risk exposure as best as they can using the available instruments offered in the market. Other aspects which makes perfect hedging undesirable could be that the construction of the portfolio may lead to a hedge which becomes too expensive. Due to competition, the life company wishes to make the guarantee attractive, and therefore is willing to take on some of the risk by only hedging part of the exposure.

Utility indifference pricing is a tool to price options in incomplete markets based on utility optimization. The indifference price is defined at the level where the issuer of the option is indifferent between entering the market on its own, or issuing the option and entering the market with the collected premium. The two investment problems are solved using stochastic control theory, and the difference between the two optimal investment plans gives the utility based hedging strategy. This way of pricing claims was first introduced by Hodges and Neuberger in their seminal paper [7]. Later indifference pricing has been analysed by many authors, see e.g. El Karoui and Rouge [3].

The indifference price will provide the lowest price for which the life company is willing to issue such guarantees. Furthermore, it is of importance for the company to know the residual risk exposure after hedging, defined as the difference between the hedge and the payoff of the guarantee. This will depend on the company's risk tolerance (given by the utility function). One may object that it is difficult to assess the risk tolerance of the company, but one may do this indirectly through deriving the Value-at-Risk (VaR) levels given by the residual risk for different tolerances. In this way the company gets a link between the VaR of their exposure, and the corresponding price they need to charge in order to achieve this VaR. We provide some numerical examples which gives a picture of the risk exposure and the price for some concrete market situations.

In this paper we shall consider the simplest market context where indifference pricing may be used, namely the pricing of put options written on a non-tradeable asset, but where we can hedge in a correlated asset. This is a picture of the situation where the life company may use only part of the fund for hedging. The fund is interpreted as a non-tradeable asset, whereas the part of it which can be traded is modelled as a separate,

but correlated, asset. Several authors have analyzed this utility indifference problem in various contexts (see e.g., Davis [1, 2], Henderson [5, 6], Monoyios [11] and Musiela and Zariphopoulou [12]). Closely related to our analysis is the papers by Henderson [6] and Monoyios [11]. Monoyios [11] derived a perturbation expansion for the price and hedging strategy in powers of $1 - \rho^2$, where ρ is the correlation between the two assets, and applied this to analyze the hedging of stock basket options using index futures. Henderson [6] considered valuation of executive incentives in terms of call options on stocks. An explicit price and hedging strategy was derived via a Feynman-Kac representation of the solution of a Black & Scholes type partial differential equation. We extend the analysis to the context of minimum interest-rate guarantees. We show that the obtained indifference price gives rise to a price dynamics which is arbitrage-free. Furthermore, we analyse the residual risk faced by the life company after applying the utility optimal hedge to cover up for the exposure, and represent this in terms of an explicit residual risk process. We extend the numerical analysis of hedging risk of Monoyios [11] to analyze different guarantees. We observe from the numerical examples that the issuer of the guarantee should in fact hedge in assets that are *negatively* correlated with the underlying of the option, since this yields a better diversification effect than using positively correlated assets as a hedge. Basing the hedge on a negatively correlated asset reduces the indifference price, but also the residual risk.

In pricing derivatives, it is desirable to have a pricing dynamics for the derivative being arbitrage-free. In the market for interest-rate guarantees, it may be difficult to exploit such an arbitrage and one may argue that it is of no importance. However, in a competitive market the prices should be fair, which means arbitrage-free. Arbitrage-freeness of the prices is a natural property since in many countries the investors may change pension provider at any time, or even turn themselves into a pension fund manager (this may be the case for bigger pension units like counties and states, or bigger firms). An incentive to do this is created if the prices offered in the market open for arbitrage. Further, it is of importance for the life company to have a price dynamics which can form the basis for marking their liabilities to the market at all times in order to have control over their risk exposure in the current market situation.

The paper is organized as follows: In the next Section we recall some results from Henderson [6] on utility indifference pricing relevant to our context. Section 3 is devoted to showing that the indifference pricing dynamics is arbitrage-free, providing an explicit Girsanov transform making it a martingale after discounting. In the following Section we state and prove a representation for the residual risk, while in Section 5 we provide several numerical examples highlighting the theory in this paper. The last Section concludes.

2. UTILITY INDIFFERENCE PRICING AND HEDGING

Assume that we are given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ satisfying the *usual hypotheses* and supporting two independent Brownian motions B and W . We consider the same market model as in Henderson [6] and Monoyios [11], however, set in a different context. Suppose the pension fund has a value dynamics Y given by a geometric

Brownian motion

$$(2.1) \quad \frac{dY}{Y} = \nu dt + \eta \left(\rho dB + \sqrt{1 - \rho^2} dW \right).$$

The correlated tradeable asset S available to the life company for hedging is supposed to have a price dynamics

$$(2.2) \quad \frac{dS}{S} = \mu dt + \sigma dB.$$

The parameters ν, μ, η, σ are constants, the two last being positive since we interpret them as volatilities. The correlation is measured in terms of the dependence between the log-returns of the two assets, and is given by $\rho \in [-1, 1]$. The case $|\rho| = 1$ coincides with the perfectly correlated case, where we are in a complete market. When $|\rho| < 1$, the market is incomplete, in which we can not hedge any claim written on Y perfectly. In this case there exists no unique arbitrage-free price. We consider the case of an investor being short λ put options written on the non-tradeable asset Y with strike K at exercise time T . Since we are interested in pricing and hedging minimum interest-rate guarantees, the strike will typically be $K = Y(0) \exp(gT)$, where g is the guaranteed rate of return on the portfolio.

We recall the results from Henderson [6] and Monoyios [11] (appropriately re-stated to fit our context) on indifference pricing based on the utility function

$$U(x) = -\frac{1}{\gamma} \exp(-\gamma x),$$

where the risk aversion is given by $\gamma > 0$. Let θ be the cash amount invested in S , with the remaining wealth invested in the riskless asset having a rate of return r . The wealth portfolio has the dynamics

$$(2.3) \quad dX_\lambda^\theta = \theta \frac{dS}{S} + r(X_\lambda^\theta - \theta) dt,$$

where the trading strategy θ is admissible when (2.3) has a unique strong solution X_λ^θ $0 \leq t \leq T$, for the initial endowment x , and

$$\mathbb{E} [-U(X_\lambda^\theta(T))] < \infty.$$

The utility indifference price is defined as the compensation charged by the life company to be indifferent between issuing the option, or not. In mathematical terms, it is defined as p_λ^γ , being the solution of

$$V(t, x + p_\lambda^\gamma, y; \lambda) = V(t, x, y; 0)$$

where V is the indirect utility function of the investor, i.e.

$$(2.4) \quad V(t, x, y; \lambda) = \sup_{\theta} \mathbb{E} [U(X_\lambda^\theta(T) - \lambda(K - Y(T))^+) | X_\lambda^\theta(t) = x, Y(t) = y].$$

Henderson [6] shows that

$$V(t, x, y; \lambda) = -\frac{1}{\gamma} \exp \left(-\gamma x \exp(r(T-t)) - (\mu - r)^2(T-t)/2\sigma^2 \right) \times w(t, y)^{(1-\rho^2)^{-1}},$$

where

$$(2.5) \quad w(t, y) = \mathbb{E}^0 \left[\exp \left(\lambda \gamma (1 - \rho^2) (K - Y(T))^+ \right) \mid Y(t) = y \right] .$$

The expectation \mathbb{E}^0 is with respect to the minimal martingale measure, denoted Q^0 , under which (B^0, W) are two independent Brownian motion with

$$dB^0 = dB + \frac{\mu - r}{\sigma} dt .$$

Thus, the Q^0 -dynamics of S is

$$(2.6) \quad \frac{dS}{S} = r dt + \sigma dB^0 .$$

whereas the Y -dynamics becomes

$$(2.7) \quad \frac{dY}{Y} = \delta dt + \eta \left(\rho dB^0 + \sqrt{1 - \rho^2} dW \right) ,$$

for

$$\delta = \nu - \eta \rho \frac{\mu - r}{\sigma} .$$

The function w is a crucial ingredient in both the price and the optimal hedging strategy, and solves the following parabolic partial differential equation with terminal condition:

$$(2.8) \quad \partial_t w + \delta y \partial_y w + \frac{1}{2} \eta^2 y^2 \partial_{yy} w = 0, \quad (t, y) \in [0, T) \times \mathbb{R}_+$$

$$(2.9) \quad w(T, y) = \exp \left(\lambda \gamma (1 - \rho^2) (K - y)^+ \right), \quad y \in \mathbb{R}_+$$

In addition, it holds

$$w(t, 0) = \exp \left(\lambda \gamma (1 - \rho^2) K \right), \quad t \in [0, T] .$$

We note that from the theory of parabolic partial differential equations, there exists a smooth solution to this problem, in the sense that $w(t, y) \in C^{1,2}([0, T) \times \mathbb{R}_+)$, the space of twice continuously differentiable functions with respect to y , and continuously differentiable with respect to t .

From the expression of V in (2.4), we find the utility indifference price as

$$(2.10) \quad p_\lambda^\gamma(t, y) = e^{-r(T-t)} \frac{\ln w(t, y)}{\gamma(1 - \rho^2)} .$$

Moreover, the optimal hedging strategy of the put option is given by

$$H_\lambda^\gamma := X_\lambda^{\theta^\lambda} - X_0^{\theta^0} ,$$

where θ^λ and θ^0 are the optimal strategies when λ and 0 options are issued, resp. This yields,

$$dH_\lambda^\gamma = (\theta^\lambda - \theta^0) \frac{dS}{S} + r(H_\lambda^\gamma - (\theta^\lambda - \theta^0)) dt .$$

The optimal cash amount θ^λ is found to be the feedback control

$$\theta^\lambda(t, y) = \frac{\mu - r}{\gamma \sigma^2 \exp(r(T-t))} + \frac{\eta \rho y}{\sigma} \partial_y p_\lambda^\gamma(t, y) .$$

Since

$$\theta^0(t, y) = \frac{\mu - r}{\gamma \sigma^2 \exp(r(T - t))},$$

we get

$$(2.11) \quad dH_\lambda^\gamma = \frac{\eta \rho y}{\sigma} \partial_y p_\lambda^\gamma(t, y) \frac{dS}{S} + r \left(H_\lambda^\gamma - \frac{\eta \rho y}{\sigma} \partial_y p_\lambda^\gamma(t, y) \right) dt.$$

The starting value of this stochastic process at time t is seen to be

$$H_\lambda^\gamma(t) = p_\lambda^\gamma(t, y).$$

We now move on to derive an EMM Q which yields the pricing dynamics of p_λ^γ .

3. CHARACTERIZATION OF THE INDIFFERENCE PRICING MEASURE

We prove that the pricing dynamics yielded by p_λ^γ is arbitrage-free, in the sense that there exists an EMM Q such that the discounted dynamics is a Q -martingale. In order to prove this, we read off the Girsanov transform from the backward stochastic differential equation which the price dynamics satisfies, and verify that this an EMM using results by Gyöngy and Martinez [4] and Liptser and Shiryaev [8, Theorem 7.7].

The price p_λ^γ may be characterized as the solution of a backward stochastic differential equation, presented in the following Lemma:

Lemma 3.1. *The stochastic process $P_\lambda^\gamma(t) = p_\lambda^\gamma(t, S(t))$ solves the backward stochastic differential equation*

$$(3.1) \quad dP_\lambda^\gamma(t) = \left\{ rP_\lambda^\gamma(t) - \frac{1}{2} \eta^2 \gamma (1 - \rho^2) Y^2(t) e^{r(T-t)} (\partial_y p_\lambda^\gamma(t, Y(t)))^2 \right\} dt \\ + \eta Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) \{ \rho dB^0 + \sqrt{1 - \rho^2} dW \}$$

$$(3.2) \quad P_\lambda^\gamma(T) = \lambda(K - Y(T))^+.$$

Proof. One may prove this by first noting that $w(t, Y(t))$ is a martingale with dynamics

$$dw(t, Y(t)) = \eta Y(t) \partial_s w(t, Y(t)) \{ \rho dB^0 + \sqrt{1 - \rho^2} dW \}.$$

Thus, applying Itô's Formula on $p_\lambda^\gamma(t, S(t))$, we get the desired backward stochastic differential equation. \square

We remark that the representation of the price process as a solution of a backward stochastic differential equation is attained in Mania and Schweizer [9] for much more general market situations.

We know from general theory of incomplete markets that there exists no unique equivalent martingale measure, but rather a continuum of such, which again implies that there are many possible arbitrage-free pricing dynamics for derivatives. E.g., for our put option, every arbitrage-free pricing dynamics will have the form

$$p(t) = e^{-r(T-t)} \mathbb{E}_Q [(K - Y(T))^+ | \mathcal{F}_t],$$

where Q is an EMM. The next result shows that $p_\lambda^\gamma(t, Y(t))$ is an arbitrage-free pricing dynamics for the put option. Furthermore, it states the explicit form of the EMM yielding the utility indifference price:

Theorem 3.2. *There exists an equivalent martingale measure Q_λ^γ such that*

$$(3.3) \quad p_\lambda^\gamma(t, y) = e^{-r(T-t)} \mathbb{E}_{Q_\lambda^\gamma} [\lambda(K - Y(T))^+ | Y(t) = y] .$$

Moreover, the Q_λ^γ -dynamics of Y and S are given by

$$\begin{aligned} \frac{dS}{S} &= r dt + \sigma dB^0 \\ \frac{dY}{Y} &= \delta^\gamma(t, Y(t)) dt + \eta \{ \rho dB^0 + \sqrt{1 - \rho^2} dW_\lambda^\gamma \} . \end{aligned}$$

Here (B^0, W^γ) are two independent Brownian motions under Q_λ^γ , with

$$dW_\lambda^\gamma = dW - \frac{1}{2} \eta \gamma \sqrt{1 - \rho^2} e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) dt .$$

Finally,

$$\delta^\gamma(t, y) = \delta + \frac{1}{2} \eta^2 \gamma (1 - \rho^2) e^{r(T-t)} y \partial_y p_\lambda^\gamma(t, y)$$

Proof. Let us first show that if Q_λ^γ , defined as the Girsanov transform stated in the Theorem, is a probability measure, then the representation of the price as a conditional expectation holds. Changing from W to W^γ , yields the dynamics

$$dP_\lambda^\gamma = r P_\lambda^\gamma dt + \eta Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) \{ \rho dB^0 + \sqrt{1 - \rho^2} dW_\lambda^\gamma \}$$

under Q_λ^γ . Thus, $e^{-rt} P_\lambda^\gamma(t)$ is a Q_λ^γ -martingale, and we find

$$e^{-rt} P_\lambda^\gamma(t) = e^{-rT} \mathbb{E}_{Q_\lambda^\gamma} [\lambda(K - Y(T))^+ | \mathcal{F}_t] .$$

This yields the result since Y is Markov under Q_λ^γ .

It remains to prove that Q_λ^γ is a probability measure. So, using Girsanov's change of measure we have to verify that

$$\mathbb{E} \left[\mathcal{E} \left(\int_0^T u(t, Y(t)) dW(t) \right) \right] = 1 ,$$

where u is defined as

$$u(t, y) = \frac{1}{2} \eta \gamma \sqrt{1 - \rho^2} e^{r(T-t)} y \partial_y p_\lambda^\gamma(t, y) ,$$

and \mathcal{E} is the stochastic exponential. To this end we think of the two Wiener processes $W(t) = W(t, \omega_1)$ and $B(t) = B(t, \omega_2)$ being defined on a probability space

$$(\Omega_1 \times \Omega_2, \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}, P_1 \otimes P_2) .$$

Since the asset Y follows the linear stochastic differential equation (2.1) we obtain

$$Y(t) = y \cdot \exp \left(\nu t + \eta \rho B(t) + \eta \sqrt{1 - \rho^2} W(t) - \frac{1}{2} \eta^2 \right) .$$

Next we set

$$b_{\omega_2}(t, z) = u \left(t, y \cdot \exp \left(\nu t + \eta \rho B(t, \omega_2) + \eta \sqrt{1 - \rho^2} z - \frac{1}{2} \eta^2 \right) \right).$$

By a result of Gyöngy and Martinez [4] we know that the stochastic differential equation

$$dZ(t) = b_{\omega_2}(t, Z(t))dt + dW(t)$$

has a unique strong solution $Z(t)$ (with respect to P_1), if

$$b_{\omega_2} \in L_{\text{loc}}^4(\mathbb{R}^+ \times \mathbb{R}).$$

However, the latter holds since b_{ω_2} is continuous P_2 -a.e. by the properties of the solution $w(t, y)$ of (2.8). Further the continuity of b_{ω_2} entails

$$\int_0^T |b_{\omega_2}(t, Z(t))|^2 dt < \infty, \quad P_1 - \text{a.e.}$$

Thus, by appealing to a result of Liptser and Shiryaev [8, Theorem 7.7], we conclude that

$$E_{P_1} \left[\mathcal{E} \left(\int_0^T b_{\omega_2}(t, W(t)) dW(t) \right) \right] = 1, \quad P_2 - \text{a.e.}$$

This implies

$$E_{P_1 \otimes P_2} \left[\mathcal{E} \left(\int_0^T u(t, Y(t)) dW(t) \right) \right] = 1.$$

□

Note that the indifference pricing measure Q_λ^γ is dependent on the option payoff and number of put options issued, as it should since the indifference price is nonlinear. Moreover, when $\gamma \downarrow 0$, then

$$w(t, y) \rightarrow 0$$

which means that $Q_\lambda^\gamma \rightarrow Q^0$. At the same time, we know from Jensen's inequality that

$$p_\lambda^\gamma(t, y) \geq \lambda e^{-r(T-t)} \mathbb{E}^0 \left[(K - Y(T))^+ \mid Y(t) = y \right].$$

In addition, a limiting argument demonstrates that

$$\lim_{\gamma \downarrow 0} p_\lambda^\gamma(t, y) = p_\lambda^0(t, y)$$

where p_λ^0 is the price under Q^0 of λ put options. Thus, the lowest indifference price is obtained when the life company has zero risk aversion, i.e. is indifferent to risk.

Consider next the prices $p_\lambda^{\gamma+}$ and $p_\lambda^{\gamma-}$, being the prices for correlations $\pm \rho, \rho \in (0, 1)$, resp. Recall the definition of the indifference price in (2.10), and use the notation w^+ and w^- for the function $w(t, y)$ in the two cases. Observe that from the definition of w , we are taking an expectation of a function of the random variable $Y(T)$. When the correlation ρ is positive, the process Y has a drift coefficient

$$\delta^+ = \nu - \eta \rho \frac{\mu - r}{\sigma} < \nu + \eta \rho \frac{\mu - r}{\sigma} = \delta^-,$$

as long as $\mu > r$. Note that the drift for correlation $-\rho$ is given by δ^- , and since the function inside the expectation defining w is non-increasing in Y , we have $w^+ > w^-$. Thus,

$$p_\lambda^{\gamma+} > p_\lambda^{\gamma-}.$$

The reason for the cheaper price with negative correlation than positive can be traced back to the utility optimization problem. The life company facing a claim being negatively correlated with the trading (or hedging) portfolio has a much more diversified total portfolio (traded and claim merged) than in the case of positive correlation. We shall later see that the risk after hedging is also improved in favour of the company.

4. A “RISKY” DECOMPOSITION OF THE HEDGING STRATEGY

Consider the utility optimal hedging strategy H_λ^γ defined in (2.11). We can prove the following decomposition of the hedging strategy at the terminal time:

Proposition 4.1. *The utility optimal hedging H_λ^γ at terminal time is equal to*

$$(4.1) \quad H_\lambda^\gamma(T) = \lambda(K - Y(T))^+ - \eta\sqrt{1-\rho^2} \int_0^T e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) dW_\lambda^\gamma(t).$$

Proof. Since

$$e^{-rt} P_\lambda^\gamma(t) = \mathbb{E}_{Q_\lambda^\gamma} [e^{-rT} \lambda(K - Y(T))^+ | Y(t)],$$

is a Q_λ^γ -martingale, a straightforward application of Itô's Formula yields

$$e^{-rT} \lambda(K - Y(T))^+ = p_\lambda^\gamma(0, y) + \int_0^T \eta e^{-rt} Y(t) \partial_y P_\lambda^\gamma(t) \{ \rho dB^0(t) + \sqrt{1-\rho^2} dW_\lambda^\gamma(t) \}.$$

Using the definition of H_λ^γ in (2.11), we find the desired decomposition. \square

We note that the residual risk is represented via the Brownian motion W_λ^γ , which is orthogonal to the Brownian motion B^0 driving the tradeable asset. Hence, the second term in H_λ^γ measures the residual risk in employing the optimal utility hedging strategy. We may restate the hedging representation as follows:

Corollary 4.2. *The utility optimal hedging H_λ^γ at terminal time is equal to*

$$(4.2) \quad H_\lambda^\gamma(T) = \lambda(K - Y(T))^+ + \int_0^T e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) dR(t),$$

where $R(t)$ is the residual risk process

$$(4.3) \quad dR = \frac{\eta\rho}{\sigma} \left(\frac{dS}{S} - r dt \right) - \left(\frac{dY}{Y} - \delta^\gamma(t, Y(t)) dt \right).$$

Proof. This follows from using the dynamics of Y and S under Q_λ^γ . \square

The residual risk process R is the sum of excessive returns from the tradeable asset S , weighted with the volatility fraction times the correlation, and the risk-adjusted excessive returns from the non-tradeable asset. The risk-adjustment of the latter comes from the “risk-free” return $\delta^\gamma(t, Y(t))$. Furthermore, the integrand

$$(4.4) \quad e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t))$$

is recognized as the time- T value of the cash amount invested in the non-tradeable asset for a perfect hedge in the Q_δ^γ risk-neutral world. In other words, in a market where the risk-free rate of return is given by $\delta^\gamma(t, Y(t))$, and where it is possible to trade the asset Y , the perfect hedging strategy of λ put options would be given by (4.4). We note that Musiela and Zariphopoulou [12] show a similar decomposition.

The residual risk is defined as the risk exposure for the issuer of the option after hedging, which in this case becomes

$$H_\lambda^\gamma(T) - \lambda(K - Y(T))^+ = \int_0^T e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) dR(t).$$

Thus, we have a quantification of this risk as the accumulated proceedings from the “perfect hedge” with respect to the residual risk process R .

Note that a direct differentiation yields the following expression for (4.4):

$$e^{r(T-t)} Y(t) \partial_y p_\lambda^\gamma(t, Y(t)) = -\lambda \mathbb{E}^0 \left[\mathbf{1}_{\{K \geq Y(T)\}} Y(T) \frac{\exp(\lambda \gamma (1 - \rho^2)(K - Y(T))^+)}{w(t, Y(t))} \mid Y(t) \right].$$

We observe that

$$\mathbb{E}^0 \left[\frac{\exp(\lambda \gamma (1 - \rho^2)(K - Y(T))^+)}{w(t, Y(t))} \mid Y(t) \right] = 1$$

and therefore the random variable

$$\frac{\exp(\lambda \gamma (1 - \rho^2)(K - Y(T))^+)}{w(t, Y(t))}$$

may be interpreted as a scaling of the Black & Scholes delta-strategy under the minimal measure. Thus, this random variable describes exactly the adjustment necessary from the delta-hedge under Q^0 , when we want to optimally hedge with risk aversion γ .

5. EXAMPLES OF INDIFFERENCE PRICING AND RESIDUAL RISK FOR INTEREST-RATE GUARANTEES

In this Section we analyze several interest-rate guarantees from the utility indifference point of view. In order to derive indifference prices and find the residual risk, we must calculate the function $w(t, y)$ numerically. This can be done by either a Monte Carlo simulation of the Feynman-Kac representation, or a numerical solution of the partial differential equation (2.8). We have chosen the latter approach, using the built-in solver `pdepe` in Matlab for parabolic problems.

5.1. A “money-back” guarantee and correlation. We begin by considering the price under the minimal measure Q^0 , henceforth referred to as the *minimal price*, and its dependence on the correlation ρ . Consider a contract which guarantees the investor that he will get his money back after one year, i.e. an interest-rate guarantee with 0% return. This situation is relevant for Norwegian pension funds where the investor may have buffers to cover possible deficits in the fund. The buffer capital is built up in years with surplus returns over the guaranteed (usually being around 3.5%). However, new legislations enforce the manager to cover a possible negative return on the fund, irrespective of the amount of buffer capital. Since many investors have large buffers, the manager is essentially issuing an at-the-money put option, that is, a guarantee against a negative return.

Suppose the fund has value 100, and that the yearly risk-free rate of return is equal to 3.5%. The pension fund has a yearly expected (log-)return of $\nu = 8\%$, with volatility $\eta = 15\%$. The hedging asset S has a slightly lower expected return $\mu = 7\%$, with volatility $\sigma = 12\%$ indicating less risk. This situation could for instance describe that only some of the assets in the pension fund can be used for hedging, and these will have less return for less risk than the total fund.

If we could have hedged perfectly using the assets in the pension fund, the Black & Scholes price for this at-the-money put option would become $p^{\text{BS}} = 4.32$ (with volatility equal to η). In Fig. 1 we have plotted the minimal price as a function of the correlation coefficient ρ between the fund and the hedging asset. Note that the price is increasing with increasing correlation, converging ultimately towards the Black & Scholes price. The interesting observation here is that when the correlation tends to -1 , we do not see a convergence towards the Black & Scholes price, but towards a price far below. If the goal is to make the guarantee cheap, the life insurance company should search for assets that is highly negatively correlated with the fund rather than positive! In the next subsection we shall consider the risk in the hedging portfolios for situations where the correlation is both negative and positive.

Based on the assumed market parameters, the prices depicted in Fig. 1 is showing the absolute minimum that the pension fund is willing to accept as compensation for giving the guarantee. This price is also the one that will yield the highest residual risk for the company. Note that the cheapest minimal price is below 2, implying that if we can hedge in instruments which are very negatively correlated with the fund the price can be reduced by more than 50%. This is of course conditioned on that the company is willing to accept the associated residual risk, which is the topic for analysis in the next subsection.

5.2. The risk in issuing “money-back” guarantees. Let us consider the same choice of parameters as above, however, supposing that the correlation is $\rho \pm 0.9$ and the risk aversion being equal to $\gamma = 0.5$. In Fig. 2 we have plotted histograms of the residual risk for the two different choices of correlation, that is, we have plotted the utility-based hedging strategy less the guarantee given at terminal time. The price for hedging with negative correlation is 3.49, considerably less than for positive correlation, being 7.32. Observe that these two prices are on both sides of the Black & Scholes price. If we have a possibility in hedging using positively correlated assets, then the guarantee becomes more expensive

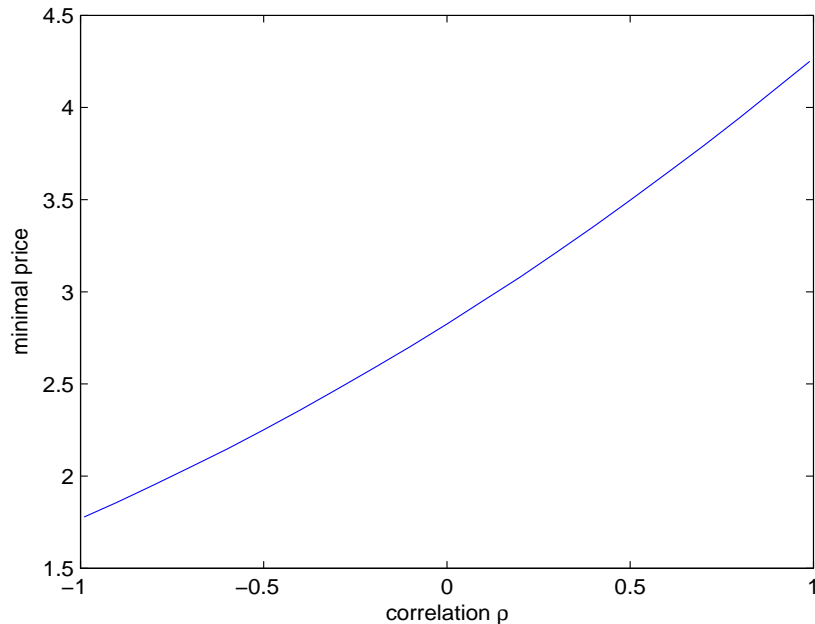


FIGURE 1. The minimal price as a function of the correlation coefficient ρ .

than the Black & Scholes, while negatively correlated hedging portfolio gives a price for the guarantee being significantly below.

We see from the histograms in Fig. 2 that the risk is less in the sense of variation for the negative correlation case, in line with the argument that there is more diversification for negative correlation than for positive. In fact, the standard deviation of the residual risk for negatively correlated assets is 2.78, compared with 3.28 for the positively correlated case. The mean of the residual risks are 2.02 and 2.90, for the negative and positive cases, respectively. Although we may earn more money on average using a positively correlated asset for hedging, the risk of losing is higher. In fact, the 1% quantile for the positively correlated case is -4.94, while the negative case has a slightly less value of -4.51. The results are based on 10.000 Monte Carlo simulations of the hedging portfolio and the pension fund, supposing 252 trading days in the year and daily updating of the hedge. Of course, the mathematically correct utility-based hedging strategy requires continuous trading, however, we believe that daily updating of the hedge gives an approximately correct picture (also in respect of the practical issues concerning continuous hedging).

For comparison, we considered a case where the correlations were $\rho \pm 0.99$. In Fig. 3 the residual risks are depicted. The prices of the option became 1.95 for the negative case, and 4.53 for the positive, in both cases a significant decrease in price as expected since we have much stronger correlation. We observe that the negatively correlated case again is much less risky, and the standard deviation is now 0.80, compared to 0.98 for the positive case.

The mean risk is 0.23 for the positive case, while it is 0.21 for the negative case. The 1% quantiles are -2.25 and -1.98, with the positively correlated case being the most risky.

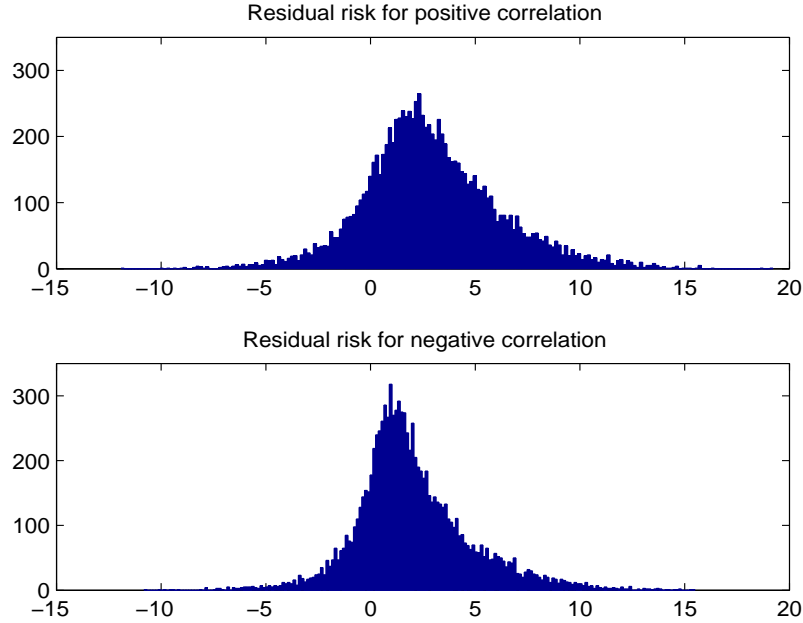


FIGURE 2. Histograms of the residual risk for a “money-back” guarantee with correlations $\rho = \pm 0.9$ and risk aversion equal to $\rho = 0.5$.

5.3. Minimum interest-rate guarantees. Let us consider an example where the life company guarantees 3.5% rate of return, while the risk-free rate of return in the market is only 2%. Norway experienced such a market situation in 2004/2005. Furthermore, we consider a life company with a lower risk profile on their pension fund than above. Suppose that the expected (log-)return is $\eta = 5\%$, with volatility being $\eta = 7\%$, and initial investment is 100. Further, the tradeable assets has a return $\mu = 6\%$, with a higher volatility of $\sigma = 10\%$. This maybe the situation for a pension fund with a large position in long bonds having a significant higher return than the short-term bills (which should be close to the risk-free rate of return). Such a position increases the portfolio return, but not necessarily the volatility if this position is not dynamically changed, but locked in. We suppose a correlation being $\rho = \pm 0.9$, and note that the Black & Scholes price becomes 3.60 based on the fund’s volatility, $\eta = 7\%$. The minimal price is equal to 3.32 for the positively correlated case, while it becomes 1.26 when the pension fund and the hedgeable assets are negatively correlated. Note that both prices are less than the Black & Scholes price.

Consider first the case with positively correlated hedging portfolio S . Supposing that the life company has a risk aversion equal to $\gamma = 0.5$, we find the indifference price being

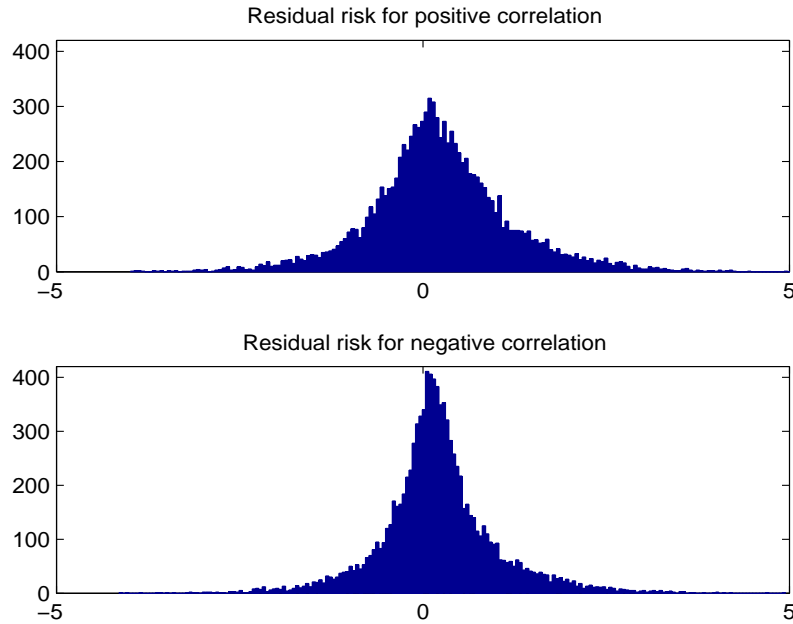


FIGURE 3. Histograms of the residual risk for a “money-back” guarantee with correlations $\rho = \pm 0.99$ and risk aversion equal to $\rho = 0.5$.

4.42, significantly higher than the corresponding minimal price and the Black & Scholes price. The 1%-quantile of the residual risk becomes -3.97 . The histogram of the residual risk is plotted in Fig. 4 (the diagram on the top). For the negatively correlated case the indifference price becomes 1.73 , again significantly higher than the minimal price, but still below the Black & Scholes price. The 1%-quantile of the residual risk now is -3.49 , which is lower than in the positively correlated case. Thus, we see again the same conclusion that it is better to hedge in negatively correlated assets from a risk point of view, but also from a pricing perspective if the goal is to have attractive prices. The histogram of the residual risk is plotted at the bottom of Fig. 4. The two histograms show a better concentration of the residual risk in the negatively correlated case, however, on average we earn more than with the positively correlated case. Another justification of this comes from the fact that the 5%-quantiles of the residual risks are -1.86 (negative) and -2.15 (positive), while the 20%-quantiles are equal. Thus, for all quantiles lower than the 20%, the negative correlated case is better from a risk perspective. The mean hedge is 0.94 (positive) and 0.59 (negative).

It is hard to determine a life company’s risk aversion, and therefore it may seem difficult to pin down a price for the guarantee using indifference pricing. However, we may use the information on the riskiness of the hedging strategy to get information about the company’s risk aversion. Deriving numerically prices and hedging strategies for many risk aversions, we will get the quantiles for the residual risk at each risk aversion level γ . The

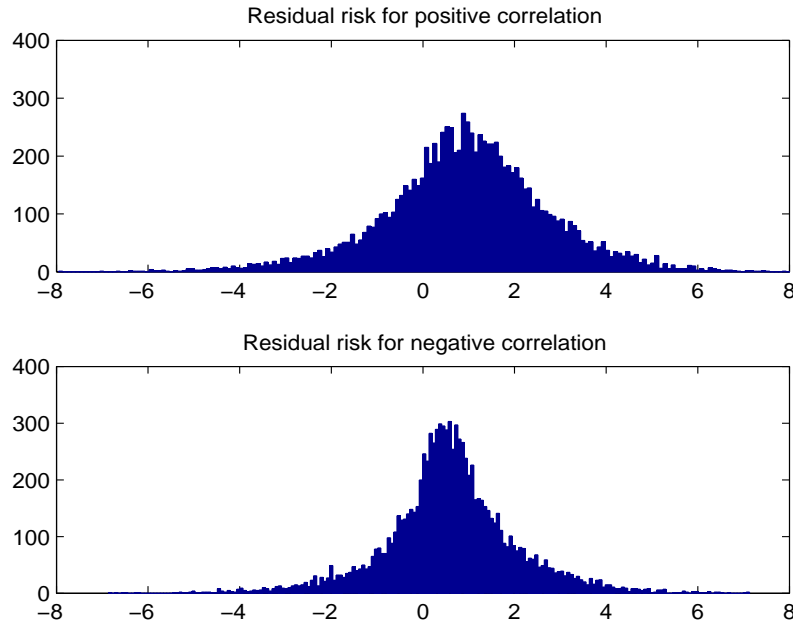


FIGURE 4. Histograms of the residual risk for a 3.5%-return guarantee with correlations $\rho = \pm 0.9$ and risk aversion equal to $\rho = 0.5$.

quantiles of the residual risk describe in reality the Value-at-Risk (VaR) levels of the life company. Thus, we can read off which risk aversion gives the desired level of VaR on the guarantee. This, in turn, provides us with the exact price and hedging strategy to use to achieve this level. Even more, the pricing tool can be used to check the residual risk exposure of the life company for given prices, set for instance by the market.

Finally, let us briefly discuss if this is a good deal for the client. We note that the risk-free return is very low, and far below the guaranteed return, and the only way the life company may achieve the return is to compose a volatile portfolio such that the expected return is higher. However, it may not seem to be a good deal for the client to pay 4.42 today in order to be assured to get back 3.5 in one year. This is indeed the case when the correlation is positive. In fact, the effective return guarantee is -0.88% in this case. For the negative correlation case, the picture is far better, namely an effective guaranteed return of 1.74% . However, this picture must be contrasted with the fact that it is difficult to achieve the high guarantee in a low-interest rate regime, and the life company possesses a high degree of risk. In Fig. 5 we have plotted the histograms of the positive and negative correlation cases for the effective return of the investor, that is, the return on the investment where we take into account the guarantee price as well. We note that there is a very high probability in both cases that the guarantee will be exercised, e.g. close to 45% risk that the investor does not get more than the guaranteed return. This figure does not coincide with the risk of the life company for having to fulfill the guarantee, because in many of these cases they

can cover all deficit using the hedge. In fact, the chance that the life company needs to raise capital to cover up for deficit in the return, after using the hedge, is 26% (positive) and 28% (negative). In addition, the hedge is removing a lot of risk in the cases where the company needs to put up additional funding to cover a return deficit. The expected

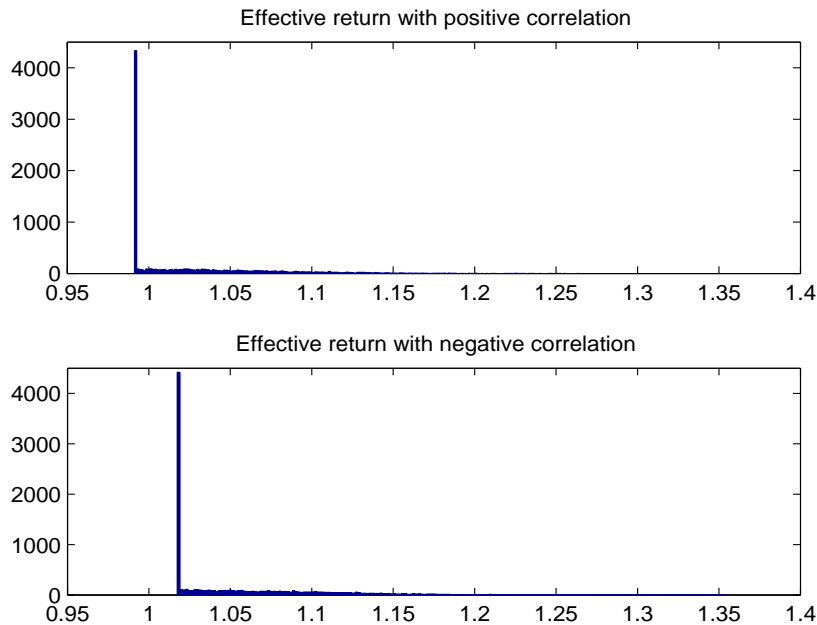


FIGURE 5. Histograms of the effective return for a 3.5%-return guarantee with correlations $\rho = \pm 0.9$ and risk aversion equal to $\rho = 0.5$.

effective return for the investor is 2.72% (positive) and 5.48% (negative). Further, there is a 25% chance that the investor gets more than 5.26% return with positive correlation, while 8.17% otherwise. The chance of getting less than 3.5% effective return is 67% (positive) and 53% (negative). Thus, there is a significant risk that the investor effectively gets less than the guarantee. Of course, these results depend heavily on the market assumptions, but still give a picture of the high risk involved for both parties of the contract.

6. CONCLUSIONS

The utility indifference pricing technique provides us with a framework for pricing and hedging an interest-rate guarantee in the case when it is not possible to hedge perfectly in the underlying portfolio. The technique requires knowledge of the risk preferences of the issuer, however this can be transformed into a question of VaR-levels for the residual risk. In this paper we have studied this technique, and proved that the prices obtained has an arbitrage-free dynamics. Moreover, we have shown a representation for the residual risk, that is, the risk exposure after using the hedge.

Numerical examples for different pension funds with return guarantees have been considered, and one of the main findings is that there is an advantage both for the life company and for the client that the hedging takes place in assets which are negatively correlated with the pension fund. This is explained by the diversification effect when we can use a hedge being negatively correlated with the underlying. We also observe that the guarantees can be made cheaper than the Black & Scholes price, and that the risk for the company, even after hedging, is large (at least in our examples).

To further increase the realism in our studies, one should consider multi-period guarantees. The pension funds are usually managed for more than 30 years, with a minimum return guarantee for each year. The interest in such a study lies in the inclusion of a dynamic buffer capital process. Each year the buffer capital is either built up according to some sharing of a return surplus, or used to partially cover up for a return deficit. The buffer capital belongs to the client, however, is part of the safety net of the life company to avoid using their owner's capital for covering a deficit. The question is to find the yearly price structure for such a contract, and how the risk is met with hedging, alongside with the buffer capital.

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